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LETTER TO THE EDITOR**Aspects of sine-Gordon solitons, defects and gates****E Corrigan and C Zambon**

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Abstract

It was recently noted how the classical sine-Gordon theory can support discontinuities, or ‘defects’, and yet maintain integrability by preserving sufficiently many conservation laws. Since soliton number is not preserved by a defect, a possible application to the construction of logical gates is suggested.

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1. Introduction

In two recent articles [1, 2], based principally on the affine Toda series of integrable models, of which the sinh-Gordon (or sine-Gordon) model is the simplest example, it was pointed out that integrable field theories in $1 + 1$ dimensions allow internal boundary conditions which preserve integrability.

Typically, at an internal boundary the classical field will have a discontinuity, hence the name ‘defect’, yet energy (and momentum) are conserved after they have been suitably modified to take into account the energy (and momentum) stored in the defect. Actually, this is already surprising given that translation invariance is destroyed by placing a defect in a particular location. Moreover, integrability is maintained in the sense that it is possible to construct a Lax pair incorporating a defect which will guarantee (and indeed generate explicitly) an infinity of other independent conserved quantities. The properties of a single defect can be repeated since any number of defects may be placed along the x -axis, each bringing an additional parameter to the model. In [1] it was pointed out how a single soliton solution is affected when the soliton encounters a defect. In this letter it is intended to expand upon this observation and point out a possible use. Naturally, it must be said at the outset that this work is entirely theoretical and speculative since, although it appears to be the case that sine-Gordon or other solitons can appear within special systems, for example Josephson junctions, polymers, or liquid crystals (see, for example, [3–5] and references therein), it is

not yet known how the specific defect introduced and described mathematically in [1] might be realized in a genuine physical system.

Solitons have remarkable properties (there are many reviews, but see for example [6, 7]) and many uses have been proposed in different contexts. In the arena of information theory or computation several ideas suggest themselves. For example, a soliton is stable and therefore might be used reliably to transport a bit of information. The sine-Gordon equation, being relativistic, requires that solitons have a maximum speed above which they cannot travel, and a train of separated solitons all moving close to that speed might be used reliably and efficiently to convey data. More remarkably, sine-Gordon solitons have a scattering property, in the sense that a fast soliton behind a slow soliton will inevitably overtake it but neither will lose their integrity. All that happens is a relative delay (a shift forwards for the faster soliton and a shift backwards for the slower one). One could imagine making use of this property also in a scenario where data are transported relatively slowly and a fast soliton might be used to overtake the data and signal to the receiver information concerning the data (that it should be ignored, for example). Besides solitons, there are also anti-solitons and these scatter maintaining their integrity not only with other anti-solitons but also with solitons. In particular, a soliton will not annihilate an anti-soliton. A soliton and an anti-soliton can make a permanently bound state (a 'breather') with centre of mass energy $0 < E < 2M_s$, where M_s is the mass of the soliton/anti-soliton.

One idea which has appeared in the literature, concerning the use of solitons to perform logical operations, seems radical because it is envisaged that the whole computation takes place using the dynamics of certain solitons (not necessarily those associated with the sine-Gordon model), dispensing entirely with standard gates [13]. The purpose of this letter is to point out that conventional computing might be carried out, albeit theoretically, by using solitons and anti-solitons to carry data, capitalizing on the properties of defects to construct standard logic gates.

2. Defects in the sine-Gordon model

First, the defect idea introduced in [1] will be reviewed briefly, following the notation and conventions established previously, and it is enough in the first instance to consider a single defect located at $x = 0$.

For convenience, the field in the region $x > 0$ will be denoted ϕ_2 and the field in the region $x < 0$ will be denoted ϕ_1 . Then, the field equations in the two regions together with the defect conditions at $x = 0$ are

$$x < 0 : \quad \partial^2 \phi_1 = -\sin \phi_1, \quad (2.1)$$

$$x > 0 : \quad \partial^2 \phi_2 = -\sin \phi_2, \quad (2.2)$$

$$x = 0 : \quad \begin{aligned} \partial_x \phi_1 - \partial_t \phi_2 &= -\sigma \sin \left(\frac{\phi_1 + \phi_2}{2} \right) - \frac{1}{\sigma} \sin \left(\frac{\phi_1 - \phi_2}{2} \right) \\ \partial_x \phi_2 - \partial_t \phi_1 &= \sigma \sin \left(\frac{\phi_1 + \phi_2}{2} \right) - \frac{1}{\sigma} \sin \left(\frac{\phi_1 - \phi_2}{2} \right). \end{aligned} \quad (2.3)$$

As pointed out in [1], these equations follow from a simple Lagrangian description of the sine-Gordon model with a defect though there is no need to review that aspect here. The form of the defect condition (2.3) is dictated by a desire to maintain integrability and it has been remarked already [1] how similar it is to a Bäcklund transformation [8]. Usually, a Bäcklund transformation relates two different solutions to a nonlinear field equation (or solutions to two

different field equations) defined in a common domain. However, in the present context the two spatial derivatives are frozen at the location of the defect. The parameter σ is free and associated with the defect. If σ is set to zero, the two fields on either side of $x = 0$ are forced to have the same value at $x = 0$ and the defect disappears. Generally, for other choices of σ there will be a discontinuity since $\phi_1(0, t) \neq \phi_2(0, t)$. If there are several defects then each will introduce its own free parameter.

An important feature of a Bäcklund transformation is its ability to generate (or remove) solitons and in fact this is one method of constructing multi-soliton solutions by repeated application—see, for example, [6, 7]. The question arises concerning the extent to which this property survives when the x -derivatives are frozen, as they are in the above defect condition. If it does, then a defect has the potential to act as a filter, or ‘gate’, at least for suitable choices of the parameter σ .

Consider first a single soliton approaching a defect at $x = 0$ from the right ($x > 0$). A convenient expression for the soliton solution in the two regions has the form (see, for example, [9, 10])

$$e^{i\phi_a/2} = \frac{1 + E_a}{1 - E_a}, \quad E_a = e^{\alpha_a x + \beta_a t + \gamma_a}, \quad \alpha_a^2 - \beta_a^2 = 1, \quad a = 1, 2, \quad (2.4)$$

with α and β real, and $\text{Im } \gamma = i\pi/2$. In order to be able to satisfy the conditions (2.3) the time dependence must match in the two domains (implying $\beta_1 = \beta_2$) and the constants γ_1, γ_2 are related by

$$\gamma_1 = \gamma_2 - \ln \left(\frac{e^\theta + \sigma}{e^\theta - \sigma} \right), \quad (2.5)$$

where it is convenient to define $\alpha_1 = \alpha_2 = \cosh \theta$ and $\beta_1 = \beta_2 = \sinh \theta$ (i.e., the soliton velocity is $-\tanh \theta$). In other words, one effect of the defect is to delay or advance the soliton as it passes through.

Suppose σ is chosen to be positive with $\sigma > 1$, then there are several interesting features to observe.

- (a) The incoming soliton solution satisfies $e^\theta > \sigma$. In this case, the soliton is delayed, though by less the faster it goes; solitons at their limiting speed ($\theta \rightarrow \infty$) are not delayed at all.
- (b) The incoming soliton satisfies $e^\theta = \sigma$. In this case, the soliton is infinitely delayed—or swallowed—by the defect. This feature was already pointed out in [1].
- (c) The incoming soliton satisfies $e^\theta < \sigma$. In this case, the delay acquires an imaginary part $i\pi$, indicating that the character of the solution ϕ_1 has changed. In fact, if ϕ_2 is a soliton then ϕ_1 is an anti-soliton, or vice versa.
- (d) A soliton travelling in the opposite direction (θ replaced by $-\theta$) will not be swallowed by the same defect as at (b). In this case, a fast soliton will be delayed and converted to an anti-soliton; with $\sigma > 1$ it will never be absorbed.

If $\sigma < 1$ the story is similar except the roles of the soliton and anti-soliton interchange.

The properties (b) and (c) are surprising, especially if one is used to the idea of topological charge—or soliton number—being conserved. On the other hand, once there is a defect, there is no longer any reason to expect soliton number to be preserved. Indeed, integrating the density for topological charge gives

$$Q = \int_{-\infty}^0 dx \partial_x \phi_1 + \int_0^{\infty} dx \partial_x \phi_2 = \phi_2(\infty, t) - \phi_1(-\infty, t) + \phi_1(0, t) - \phi_2(0, t), \quad (2.6)$$

and it becomes clear the difference $\phi_1 - \phi_2$ measures the strength of the defect at $x = 0$. Effectively, in cases (b) or (c), respectively, the defect is storing one or two units of

topological charge. These two phenomena also fit well with the traditional uses of Bäcklund transformations.

The different effects of the defect on solitons moving in opposite directions is reflected in an intriguing feature of (2.3) with respect to its behaviour under time reversal. In each of the bulk regions the sine-Gordon equations are invariant separately under the transformations, $t \rightarrow -t$ and $\phi_a \rightarrow -\phi_a$, $a = 1, 2$. In contrast, the defect condition is invariant only under the combinations of any pair of these, for example $t \rightarrow -t$ and $\phi_1 \rightarrow -\phi_1$ or $\phi_2 \rightarrow -\phi_2$, together with $\sigma \rightarrow 1/\sigma$. For a given choice of σ this implies the model loses its time-reversal invariance, meaning there is no time-reversed process to (b) in which, for example, a defect might emit a soliton.

If several solitons approach a defect then they interact with it independently of one another, each being delayed. This is not difficult to check using an explicit two soliton solution of the form [9]

$$e^{i\phi/2} = \ln(\tau_0/\tau_1), \quad \tau_p = 1 + (-)^p (E^{(1)} + E^{(2)}) + A_{12} E^{(1)} E^{(2)}, \quad p = 0, 1$$

$$A_{12} = -\tanh^2 \left(\frac{\theta_1 - \theta_2}{2} \right) \quad (2.7)$$

$$E^{(a)} = e^{\alpha^{(a)}x + \beta^{(a)}t + \gamma^{(a)}}, \quad a = 1, 2$$

in each of the two regions, and imposing the boundary condition (2.3). The defect condition requires

$$\gamma_1^{(1)} = \gamma_2^{(1)} - \ln \left(\frac{e^{\theta_1} + \sigma}{e^{\theta_1} - \sigma} \right) \quad (2.8)$$

$$\gamma_1^{(2)} = \gamma_2^{(2)} - \ln \left(\frac{e^{\theta_2} + \sigma}{e^{\theta_2} - \sigma} \right). \quad (2.9)$$

Note, by adding $i\pi$ to any of the constants $\gamma^{(a)}$, either one or other, or both, soliton components can be converted to anti-solitons.

An interesting and important point to note is that the defect can absorb at most one soliton (or anti-soliton), given a suitable σ , but not both because a genuine two soliton solution requires $\theta_1 \neq \theta_2$. On the other hand, neither, or one, or both may be converted to a soliton of the opposite character according to the relative magnitudes of e^{θ_1} , e^{θ_2} and σ . These observations appear to suggest that a defect might be used to model logic gates.

3. The defect as a logical gate

In this section it will be supposed that $\sigma > 1$ and solitons approach a defect from the right ($x > 0$).

Adopting the convention that a soliton represents *true* or '1', and an anti-soliton *false* or '0', the simplest gate to model is NOT since it is enough that the soliton (or anti-soliton) approaching the defect is moving slowly with $e^\theta < \sigma$.

Using the defect to remove a soliton or anti-soliton is not by itself enough to reproduce the full variety of logic gates. For example, if the first soliton to reach the gate is removed, the second must be travelling slower and will be inverted. This is illustrated in table 1 where the first soliton to arrive at the defect is labelled a_1 and the second a_2 . On the other hand, if the second arrival is to be removed the first will pass the defect delayed yet retain its character. Neither of these is especially useful but together they exhaust the possibilities with a passive defect.

Table 1. Sample gate using a passive defect.

a_2	a_1	$a_2 * a_1$
1	1	0
1	0	0
0	1	1
0	0	1

Table 2. XOR gate using an active defect.

a_2	a_1	$a_2 \text{ XOR } a_1$
1	1	0
1	0	1
0	1	1
0	0	0

Table 3. NAND gate using an active defect.

a_3	a_2	a_1	$a_2 \text{ NAND } a_1$	a_2	a_1
1	1	1	0	1	1
1	1	0	1	1	0
1	0	1	1	0	1
1	0	0	1	0	0

On the other hand, if it is supposed there is feedback, which allows the passage of a soliton (but not an anti-soliton) to initiate a signal instructing the defect ‘controller’ to raise the defect parameter, then the possibilities become more interesting. For example, it would be possible to arrange the first arrival to be removed and then raise the defect parameter if it is a soliton, but merely to be removed if it is an anti-soliton leaving the defect parameter unchanged. Under such circumstances, the second arrival will be inverted if the first arrival is a soliton but not if it is an anti-soliton. This allows, in the same notation as that used in table 1, an XOR gate to be represented (table 2).

However, even with this device it is clear the output will always have an even number of ‘1’s and therefore both NOR and NAND (from which all other standard two-bits-in-one-bit-out gates can be constructed) are unobtainable. Besides, a careful tuning of the defect parameter is required to remove a soliton and it might be better to allow both solitons to pass, adjusting the defect parameter each time there is a passing soliton, but not for a passing anti-soliton. With two bits this will just reproduce the XOR gate already described.

On the other hand, consider a triple of approaching solitons/anti-solitons. Each will be affected by the defect independently of the others (checked as before in the case of two using Hirota’s explicit solution), and arrange for each passing soliton (but not anti-soliton) to increase the defect parameter by the same amount δ with

$$e^{\theta_1} > \sigma, \quad e^{\theta_2} > \sigma + \delta, \quad \sigma + 2\delta > e^{\theta_3} > \sigma + \delta.$$

The three arriving solitons/anti-solitons have rapidities $\theta_1 > \theta_2 > \theta_3$. With this arrangement, the first pair of solitons/anti-solitons pass the defect, delayed but without inversion, but the third is inverted provided the first two were both solitons, and not otherwise. If the third is always a soliton this provides a version of the NAND gate for the first two (table 3). In fact, using all possible triples in and out with the rules described above gives a representation of the Toffoli three-bit gate [14].

4. Discussion

As stated in the introduction, it is not yet clear if there is a set of physical circumstances permitting the type of integrable defect whose properties have been discussed. The search for such a system continues. It would also be interesting to investigate further the quantum aspects of field theories with defects (developing ideas originally pioneered by Delfino *et al* [15]). In the past, the quantum sine-Gordon model has been investigated (see, for example [16]), mostly from an algebraic point of view, but it is not yet clear all the properties outlined above (and those given in [1, 2]) have been properly taken into account. In particular, no mention has been made previously concerning a defect's ability to change topological charge by ± 1 , although it is clear the transmission matrix discovered by Konik and LeClair does allow transitions between solitons and anti-solitons changing topological charge by ± 2 . Perhaps the transmission matrix constructed by them will need to be generalized, or perhaps the capacity of a defect to remove a soliton/anti-soliton does not survive quantization; it is, after all, a delicate matter since the incoming soliton rapidity needs to be precisely matched to the defect parameter, yet quantum effects are rarely so sharply tuned.

Some time ago, Baseilhac and Delius discovered dynamical boundary conditions for the sine-Gordon model restricted to a half-line [17]. Seeking the analogues of these in association with a defect may prove profitable.

One might also wonder about the possibility of finding representations of two-qubit gates or a three-qubit gate, such as the generalization of the Toffoli gate due to Deutsch [18], using solitons within an integrable quantum field theory.

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